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Solutions for supersonic rotational flow around a corner using a new co-ordinate system

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A co-ordinate system consisting of the left-running characteristics (α = const.) and the streamlines (ψ = const.) is used. The governing equations are derived in terms of α and ψ for a two-dimensional steady supersonic rotational inviscid flow of a perfect gas. The equations are applied to the problem of an initially parallel supersonic rotational flow which expands around a convex corner. The velocity of the incoming flow at the wall is considered to be either supersonic (case 1) or sonic (case 2). For each case, solutions uniformly valid in the region near the leading characteristic and in the region near the corner, are found for the Mach angle and flow deflexion angle in terms of their values on the leading characteristic and at the corner. In case 2, a transonic similarity solution is found and composite solutions are constructed for each region. Comparisons are made with existing exact numerical results.

1. Introduction

Analytical solutions for supersonic rotational flows have been limited, for the most part, to those problems where the effects of rotation are small enough that they may be considered as perturbations from a known irrotational solution. If the usual Cartesian co-ordinates are used, then the characteristics are those calculated from zeroth-order irrotational flow conditions. Thus, a signal is propagated into the flow along an irrotational flow characteristic instead of along one which has been adjusted to take account of the rotation; far from the disturbance, the cumulative effects of this error can be such as to invalidate the solution. The simple perturbation theory is, therefore, not uniformly valid throughout the field and more sophisticated techniques must be employed to make it so (Van Dyke 1964).

For supersonic flows with large rotation, numerical solutions are obtained by employing the method of rotational characteristics. In this method, it is necessary to compute the values of the entropy and total enthalpy which exist along the streamline which passes through the intersection of two characteristics at a given net point. The accuracy with which these computations are made can be improved by using iteration methods at each point (Ferri 1954). The calculations result in a knowledge of the physical location of the two families of characteristics and the flow properties at each intersection; the location of a given streamline must be found by another calculation.

In this paper, a set of independent variables is proposed which is different from that used previously in either analytical or numerical computations for supersonic flows. This set is composed of α and ψ, where α is constant along a left running characteristic and ψ is constant along a streamline. These co-ordinates are not orthogonal and thus do not have the geometric simplicity of orthogonal curvilinear co-ordinates. Moreover, because the two-length metric coefficients are different, they must be retained in the equations, a condition which does not arise in the method of characteristics. On the other hand, the proposed co-ordinate system has the virtue that approximate solutions found in terms of α and ψ do not contain any inherent order of approximation for the shape of the characteristics. Perturbation solutions may be found in terms of α and ψ and the proper forms of the left running characteristics and streamlines are found after these solutions have been obtained, to the same order of approximation. Thus, according to Whitham (1952), "...linearized theory gives a valid first approximation to the flow everywhere provided that in it the approximate characteristics are replaced by the exact ones, or at least by a sufficiently good approximation to the exact ones", and here, the co-ordinate system has been set up such that the solutions are found in terms of the exact left-running characteristics. It is clear that this co-ordinate system is especially useful when initial conditions are known along a leading characteristic and boundary conditions are given along a streamline.

The problem considered in this paper is that of two-dimensional rotational supersonic flow around a convex corner. The flow is assumed to be steady and inviscid and to be composed of a perfect gas with constant specific heats. The incoming flow, that is, the flow which enters the expansion region across the leading characteristic, is assumed to be parallel flow with either a supersonic, or sonic Mach number at the wall. The former case should be useful in the analysis of the flow at the base of an ogive-cylinder supersonic body, while the latter should be useful in approximating the expansion of a supersonic boundary layer around the corner at the base of a re-entry vehicle. It is well known that if a supersonic boundary layer is expanded around a corner with a significant pressure decrease, the pressure forces are large compared to the viscous forces, and the turning may be described by inviscid equations. The flow entering the expansion region thus has gradients in the flow variables and is rotational. It has been shown recently by Olsson & Messiter (1968) that in a hypersonic boundary layer the flow is not quite parallel as it enters the expansion region; instead, the streamlines converge slightly. In addition, the assumption made here that the expansion is centred on the initially sonic streamline means that the subsonic portion of the boundary layer is neglected. Nevertheless, this model of the expansion of the boundary layer at a corner has been shown to be an excellent approximation by several authors (Weinbaum 1966; Weiss & Weinbaum 1966; Weiss 1967; Weiss & Nelson 1968).

In the next section, the governing equations are derived in the new co-ordinate system and in subsequent sections, solutions are presented for various regions of interest. Details of the calculations presented here may be found in the reference (Adamson 1968).

2. Derivation of equations

In the following, the dependent variables are made dimensionless with respect to reference values, with the exception of the entropy and total, or stagnation, enthalpy. The entropy is made dimensionless with respect to the specific gas constant, R, and the total enthalpy with respect to the square of the reference velocity, $\bar{q}_r$. All lengths are referred to a characteristic length L. Dimensional quantities are denoted by a bar.

The co-ordinate system and the associated notation are shown in figure 1. There, r and ϕ are polar co-ordinates, $\mu = \sin^{-1}(1/M)$ is the Mach angle, and θ is the inclination of a streamline from the horizontal. The velocity components are shown in figure 2; u_α and v_α are the velocity components parallel and perpendicular to the left-running characteristics, respectively, and u and v are the usual radial and tangential velocity components. The relationships between the various components are,

$$u = u_\alpha \sin\omega - v_\alpha \cos\omega, \tag{1a}$$

$$v = u_\alpha \cos\omega + v_\alpha \sin\omega, \tag{1b}$$

where

$$\omega = \theta + \phi + \mu. \tag{2}$$

Now since $\bar{v}_\alpha$ is perpendicular to a characteristic, it is equal to the speed of sound, $\bar{a}$. Thus, in dimensionless form,

$$v_\alpha = a/M_r, \tag{3}$$

where $M_r = \bar{q}_r/\bar{a}_r$ is a reference Mach number. Since a perfect gas is assumed, then

$$T = M_r^2 v_\alpha^2, \tag{4}$$

where T is the temperature, and the total enthalpy, h_t, may be written as

$$h_t = \frac{\beta T}{M_r^2} + \frac{u_\alpha^2 + v_\alpha^2}{2} = \frac{1}{2\Gamma^2}(v_\alpha^2 + \Gamma^2 u_\alpha^2), \tag{5}$$

where $\Gamma^2 = (\gamma-1)/(\gamma+1)$ and γ is the ratio of specific heats. Now h_t is a function of ψ alone and u_α and v_α depend on both α and ψ. Hence, (5) is satisfied in general if

$$v_\alpha = (h_{t_1})^{\frac{1}{2}} \cos\epsilon, \tag{6a}$$

$$u_\alpha = (h_{t_1})^{\frac{1}{2}} (\sin\epsilon)/\Gamma, \tag{6b}$$

$$h_t = (h_{t_1})/2\Gamma^2, \tag{6c}$$

where $h_{t_1} = h_{t_1}(\psi)$ and $\epsilon = \epsilon(\alpha, \psi)$. It should be noted that whereas h_t is a given function of ψ, ϵ is a general function of α and ψ to be determined. In Prandtl–Meyer flow, h_t is a constant and $\epsilon = \Gamma\phi + \text{constant}$; equations (6) are seen to be a simple generalization of the Prandtl–Meyer solution. Since the Mach number may be written as

$$M = M_r \frac{q}{a} = \frac{q}{v_\alpha},$$

the relationships between ϵ and μ and ν, the Prandtl–Meyer angle, are easily shown to be

$$\tan\epsilon = \Gamma\cot\mu, \tag{7a}$$

$$\nu = \mu+\frac{\epsilon}{\Gamma}-\frac{\pi}{2}. \tag{7b}$$

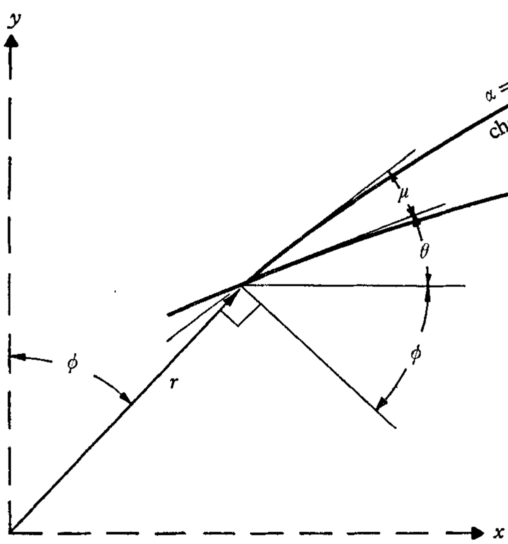

FIGURE 1. Sketch of co-ordinate systems.

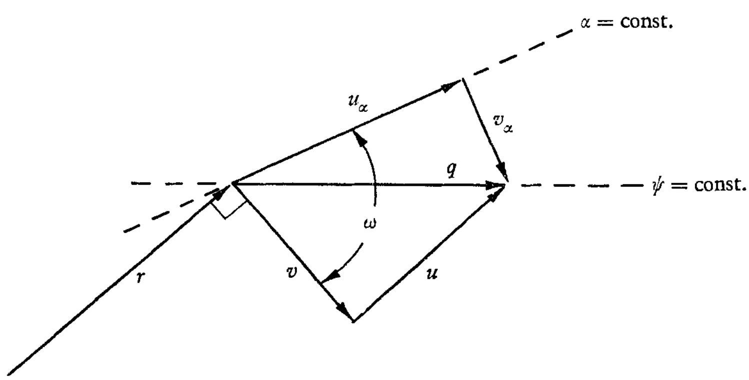

FIGURE 2. Velocity components in the (r, ϕ) and (α, ψ) co-ordinate systems.

The entropy is a known function of ψ. Hence, a known function of ψ, say F_1, may be defined as follows:

$$dS = \beta d[\ln(P\rho^{-\gamma})] = -d[\ln F_1(\psi)], \tag{8}$$

where $\beta = (\gamma-1)^{-1}$, and S, P and ρ are the dimensionless entropy, pressure and density, respectively. The pressure and density are also related by the equation of state

$$P = \rho T = \rho M_r^2 v_\alpha^2, \tag{9}$$

where (4) has been used for the temperature. Equations (8) and (9) are satisfied if

$$\rho = F_1 v_\alpha^{2\beta}, \tag{10a}$$

$$P = M_r^2 F_1 v_\alpha^{2\gamma\beta}. \tag{10b}$$

Thus, from (4), (6) and (10), the thermodynamic properties and velocity components u_α and v_α may be related to ϵ and known functions of ψ.

The governing equations may be written in terms of α and ψ with simple transformations. In intrinsic co-ordinates, the continuity and momentum equations may be combined to give the following relations (Hayes & Probstein 1966, p. 482):

$$\frac{\partial\theta}{\partial n} + \frac{\cos^2\mu}{\gamma P}\frac{\partial P}{\partial s} = 0, \tag{11a}$$

$$\frac{\partial\theta}{\partial s} + \frac{\sin^2\mu}{\gamma P}\frac{\partial P}{\partial n} = 0, \tag{11b}$$

where s denotes distance measured along a streamline and n denotes distance measured perpendicular to a streamline. Now, if α and ψ are to be independent variables such that $h_\alpha d\alpha$ and $h_\psi d\psi$ denote the elements of length on the α lines (ψ = const.) and the ψ lines (α = const.) respectively, then the local relation between intrinsic co-ordinates and the (α, ψ) co-ordinates is expressed by the differentiation formulae,

$$\frac{\partial}{\partial s} = \frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}, \tag{12a}$$

$$\frac{\partial}{\partial n} = -\frac{\cot\mu}{h_\alpha}\frac{\partial}{\partial\alpha} + \frac{1}{h_\psi \sin\mu}\frac{\partial}{\partial\psi}. \tag{12b}$$

If this transformation is applied to (11) and substitution is made for P according to (10*b*), the resulting equations may be rearranged to give:

$$\frac{\partial}{\partial\psi}\left(\mu + \frac{\epsilon}{\Gamma} - \theta\right) = \sin\mu\cos\mu\frac{dH}{d\psi}, \tag{13a}$$

$$\frac{1}{h_\psi}\frac{\partial\theta}{\partial\psi} = \frac{\cos\mu}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\mu + \frac{\epsilon}{\Gamma} + \theta\right), \tag{13b}$$

where

$$\begin{aligned} dH &= \beta d(\ln h_{t_1}) + \gamma^{-1} d(\ln F_1) \\ &= \beta d(\ln h_t) - \gamma^{-1} dS. \end{aligned} \tag{14}$$

In order to find expressions for the length metric coefficients, h_α and h_ψ, it is convenient to use the local relations between the (α, ψ) and polar co-ordinates,

$$\frac{1}{h_\psi}\frac{\partial}{\partial\psi} = \sin\omega\frac{\partial}{\partial r} + \frac{\cos\omega}{r}\frac{\partial}{\partial\phi}. \tag{15a}$$

$$\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha} = \sin(\theta+\phi)\frac{\partial}{\partial r} + \frac{\cos(\theta+\phi)}{r}\frac{\partial}{\partial\phi}. \tag{15b}$$

If (15*a*) is applied to ψ and the continuity equation in polar co-ordinates is employed to define $\partial\psi/\partial r$ and $\partial\psi/\partial\phi$, it is readily shown that

$$h_\psi = 1/\rho v_\alpha. \tag{16}$$

Next if equations (15) are applied first to r and then to ϕ, the following relations result:

$$\frac{1}{h_\alpha}\frac{\partial r}{\partial \alpha} = \sin(\theta+\phi), \tag{17a}$$

$$\frac{1}{h_\alpha}\frac{\partial \phi}{\partial \alpha} = \frac{1}{r}\cos(\theta+\phi); \tag{17b}$$

and

$$\frac{1}{h_\psi}\frac{\partial r}{\partial \psi} = \sin\omega, \tag{18a}$$

$$\frac{1}{h_\psi}\frac{\partial \phi}{\partial \psi} = \frac{\cos\omega}{r}. \tag{18b}$$

Finally, if derivatives of (17) are taken with respect to ψ, and (17) and (18) are used to rid the resulting equations of r and ϕ, they may be combined to give a relation for h_α in terms of α and ψ; with simple substitutions this relation may be written as,

$$\frac{1}{h_\psi}\frac{\partial h_\alpha}{\partial \psi} = -\sin\mu\frac{\partial}{\partial \alpha}\left(\mu+\frac{\epsilon}{\Gamma}+\theta\right)+\frac{1}{\sin\mu}\frac{\partial}{\partial\alpha}(\epsilon/\Gamma). \tag{19}$$

Since ϵ and h_ψ may be written in terms of μ and known functions of ψ, (13) and (19) may be considered as three equations for the unknowns μ, θ and h_α. Equations (17) and (18), or their equivalents, may be used to find the physical location of the left-running characteristics and streamlines in the flow field.

3. Prandtl–Meyer flows

In this section, expansion of a supersonic flow around a corner is considered for the case where H is a constant. It can be seen from (13) that when $dH/d\psi = 0$, a solution is

$$\mu+\epsilon\Gamma^{-1}+\theta = \nu+\theta+\tfrac{1}{2}\pi = \text{const.} \tag{20a}$$

$$\theta = \theta(\alpha). \tag{20b}$$

Since ϵ may be written in terms of μ (equation (7a)), then

$$\mu = \mu(\alpha), \quad \epsilon = \epsilon(\alpha). \tag{21a, b}$$

Thus, the Mach number and flow deflexion angle are constant along any left-running characteristic in the expansion region. In particular, this must be the case along the leading characteristic, so this solution holds when the incoming flow is parallel with no gradient in Mach number. Furthermore, since the equation for a left-running characteristic (α = constant) is, in Cartesian co-ordinates,

$$dy/dx = \tan(\theta+\mu), \tag{22}$$

the characteristics are radial lines. Thus, this is a simple wave solution and is, essentially, the Prandtl–Meyer solution. However, it should be noted that the fact that H is constant does not imply that the flow is necessarily irrotational. From (14), it is seen that $dH = 0$ either when h_t and S are both constants, which is the irrotational case, or when the variations in h_t and S across streamlines just balance each other, such that $\beta d \ln h_t = \gamma^{-1} dS$. In the latter case the flow is rotational, but it expands along each streamline as though it were part of a Prandtl–Meyer flow with upstream conditions being those given on the streamline in question. This solution was given previously by Cole (1965) with the derivation proceeding along somewhat different lines.

4. Asymptotic solutions for supersonic rotational flow around a convex corner

This section contains asymptotic solutions derived for two different regions. The first is the region near the leading characteristic and the second is the region near the corner. The complete region under consideration is shown as the shaded region in figure 3. In all cases, the incoming flow is assumed to be parallel.

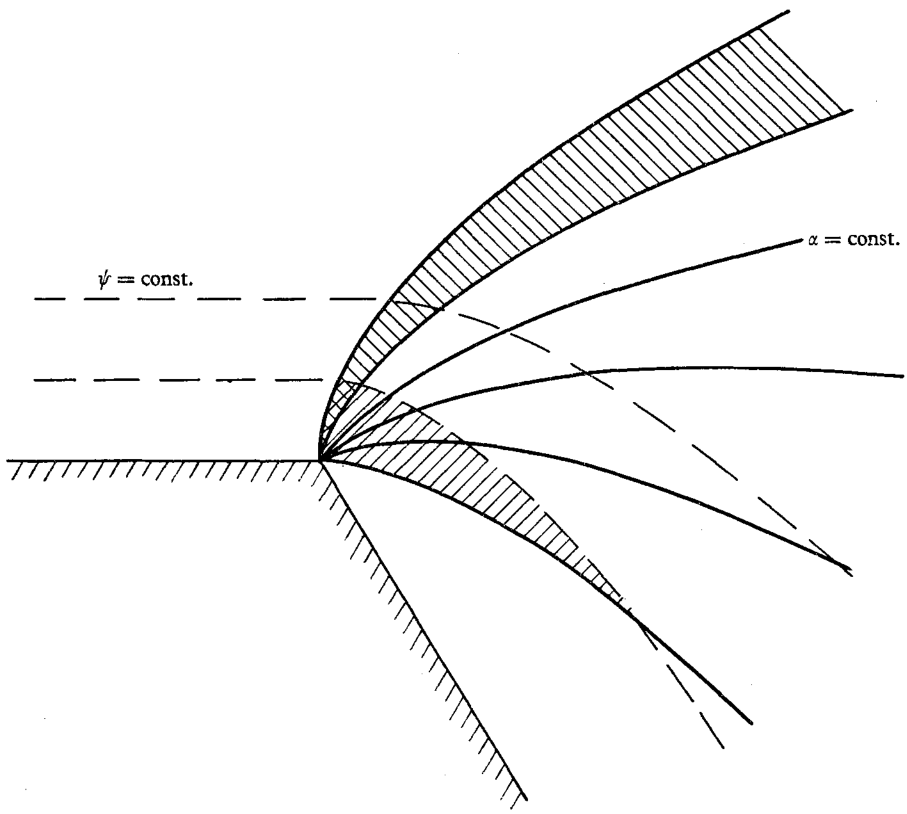

FIGURE 3. Flow regions covered by asymptotic solutions.

Two different conditions are considered for the velocity of the incoming flow at the wall; in the first, it is supersonic and in the second it is sonic. When the first condition holds, it is shown that the solutions valid in each of the regions mentioned above agree in the common region, so that a composite solution may be constructed. When the second (sonic) condition holds, it is shown that a transonic similarity solution may be found which matches with each of the solutions in the above-mentioned regions. In this case, composite solutions are found for each region.

In each of the following solutions it is assumed that at the corner, Prandtl–Meyer solutions hold, so that the left running characteristics are tangent to radial lines at the corner. Then α is associated with ϕ_0, the value of ϕ for the radial line in question, at the wall. That is,

$$\alpha = \phi_0 + \text{const.} \tag{23}$$

Henceforth, the subscript 0 is used to denote conditions at the corner, and the subscript i is used to denote conditions along the leading characteristic. The double subscript $i0$, then, is used for conditions at the point where the leading characteristic meets the corner.

Solutions valid near the leading characteristic for $M_{i0} > 1$

Since α is associated with ϕ_0 and this expansion is to be carried out for $\alpha \ll 1$, the following solutions are valid for large rotation but small turning angle. Along the leading characteristic ($\alpha = 0$), flow properties are assumed known as functions of ψ and asymptotic expansions of the following form are assumed;

$$\mu = \mu^{(0)}(\psi) + f_1(\alpha)\mu^{(1)}(\psi) + \ldots, \tag{24}$$

where $\mu^{(0)} = \mu_i$ is the given Mach angle distribution along the leading characteristic and where

$$\mu^{(n)}(\psi) = O(1)$$

and

$$f_n(\alpha) \ll 1,$$

such that

$$\lim_{\alpha \to 0} \frac{f_n + 1}{f_n} = 0.$$

Similar expansions are used for ϵ and θ. The proper form of expansion for h_α may be inferred from (19) to be

$$h_\alpha = f_1'(\alpha) h_\alpha^{(1)}(\psi) + \ldots, \tag{25}$$

where $f_1' = df_1/d\alpha$.

When the above expansions are substituted into (13) and (19), and terms of like order in $f_1(\alpha)$ are collected, the zeroth- and first-order equations are found to be, if $d\theta^{(0)}/d\psi = d\theta_i/d\psi = 0$:

$$\frac{d}{d\psi}\left(\mu_i + \frac{\epsilon_i}{\Gamma}\right) = \sin\mu_i \cos\mu_i \frac{dH}{d\psi}, \tag{26a}$$

$$\frac{d}{d\psi}\left(\mu^{(1)} + \frac{\epsilon^{(1)}}{\Gamma}\right) = \frac{d\theta^{(1)}}{d\psi} + \mu^{(1)}(\cos^2\mu_i - \sin^2\mu_i)\frac{dH}{d\psi}, \tag{26b}$$

$$\mu^{(1)} + \frac{\epsilon^{(1)}}{\Gamma} + \theta^{(1)} = 0, \tag{26c}$$

$$\frac{dh_\alpha^{(1)}}{d\psi} = h_\psi^{(0)}\left\{-\sin\mu_i\left(\mu^{(1)} + \frac{\epsilon^{(1)}}{\Gamma} + \theta^{(1)}\right) + \frac{1}{\sin\mu_i}\frac{\epsilon^{(1)}}{\Gamma}\right\}, \tag{26d}$$

where $h_\psi^{(0)} = (\rho v_\alpha)_i^{-1}$. Equation (26*a*) is simply the relationship between H and the Mach number distribution which must be satisfied along the leading characteristic. The solution to (26*b*) and (26*c*) is,

$$\mu^{(1)} + \frac{\epsilon^{(1)}}{\Gamma} = -\theta^{(1)} = (\sin\mu_i \cos\mu_i)^{\frac{1}{2}}, \tag{27}$$

where the constant of integration is included in f_1 and will be found later. With $\theta^{(1)}$ and $\mu^{(1)} + \epsilon^{(1)}/\Gamma$ known, equations (26*d*) may be solved, but this is necessary only when the physical location of the characteristics is desired, for a given problem. If the expansions for μ and ϵ are substituted in (7*a*), a relation between $\mu^{(1)}$ and $\epsilon^{(1)}$ may be found and combined with (27) to yield an expression for $\mu^{(1)}$ alone. Thus, the solutions for θ and μ are, to first order,

$$\theta = -f_1(\alpha)(\sin\mu_i \cos\mu_i)^{\frac{1}{2}}, \tag{28a}$$

$$\mu = \mu_i - \frac{f_1(\alpha)}{1-\Gamma^2}(\sin^2\mu_i + \Gamma^2\cos^2\mu_i)\frac{(\sin\mu_i)^{\frac{1}{2}}}{(\cos\mu_i)^{\frac{3}{2}}}. \tag{28b}$$

In general, $f_1(\alpha)$ may be found by evaluating either of equations (28) at the corner where $\psi = 0$, $\mu_i = \mu_{i0}$, etc. It is readily shown that the expressions are equivalent if the Prandtl–Meyer solution holds at the corner. Hence

$$\begin{aligned} f_1(\alpha) &= (-\theta_0)(\sin\mu_{i0}\cos\mu_{i0})^{-\frac{1}{2}} \\ &= (1-\Gamma^2)(\mu_{i0}-\mu_0)\frac{(\cos\mu_{i0})^{\frac{3}{2}}}{(\sin\mu_{i0})^{\frac{1}{2}}(\sin^2\mu_{i0}+\Gamma^2\cos^2\mu_{i0})} \end{aligned} \tag{29}$$

and (28) and (29) complete the solution for θ and μ. In the following expressions, these solutions are written in terms of the Mach number. In addition, (10*b*) has been used to form the term $(P-P_i)/P_i$, where P_i is the pressure of the incoming flow on the streamline which passes through the point where the pressure is P:

$$\theta = \theta_0 \frac{M_{i0}}{M_i}\left[\frac{M_i^2-1}{M_{i0}^2-1}\right]^{\frac{1}{4}}, \tag{30a}$$

$$M = M_i - \frac{\theta_0}{1-\Gamma^2}\frac{M_{i0}}{(M_{i0}^2-1)^{\frac{1}{4}}}\frac{1+\Gamma^2(M_i^2-1)}{(M_i^2-1)^{\frac{1}{4}}}, \tag{30b}$$

$$\frac{P-P_i}{P_i} = \gamma\theta_0\frac{M_{i0}}{(M_{i0}^2-1)^{\frac{1}{4}}}\frac{M_i}{(M_i^2-1)^{\frac{1}{4}}}. \tag{30c}$$

Equations (30) relate θ, M and P at a point located by a given streamline and left running characteristic, to the values of these variables both in the incoming flow on the given streamline and at the corner on the given characteristic. The dependence of θ and P on M_i (i.e. the ψ dependence) has been given by Weinbaum (1965, 1966) by a different method. For example, when considering the turning of a supersonic boundary layer, where the Mach number of the incoming flow is unity at the wall, he showed that

$$(M_i^2-1)^{\frac{1}{4}}/M_i\theta = \text{const.}$$

is a solution as long as M_i does not approach unity, i.e. near the leading characteristic but away from the corner. The constant was found by comparing the above expression with a rotational characteristic solution at a given M_i. Here, it is shown that the solution holds in general and the 'constant' may be evaluated for a rotational corner flow where $M_{i0} \neq 1$. Later, a solution is found for the case where $M_i = 1$.

It is clear that the solutions given in (30) are not valid for $M_{i0} = 1$. However, (28) should be valid outside a transonic region and should match with the transonic solution in the region near the leading characteristic. The matching procedure should thus lead to the proper form for $f_1(\alpha)$.

Solutions valid near the corner, for $M_{i0} > 1$

In this section, approximate solutions are found for the case $\psi \ll 1$, where $\psi = 0$ is associated with the wall. Since ψ is dimensionless with respect to a reference velocity, $\bar{q}_r$, and a characteristic length, L, the condition $\psi \ll 1$ corresponds to distances from the corner small compared to L. Here L is associated with the vorticity or equivalently with the gradient of H since this is the only source of a characteristic length in the problem. Hence these solutions are valid for small rotation but large turning angle, because no restriction is placed on α.

Since $r = O(h_\psi \psi)$ and $h_\psi = O(1)$, it is seen that, as mentioned above $\psi \ll 1$ corresponds to $r \ll 1$. Hence the expansions found for this case are similar to those found by Johannesen & Meyer (1950) and Pai (1954). In particular, Pai considered the special case of supersonic rotational flow around a corner for iso-energetic flows. The solutions given here are more general in that both total enthalpy and entropy variations are allowed; more importantly they are found later for the case where the incoming flow at the wall is sonic, a problem which has not been considered heretofore. Finally, the solutions are shown to be independent of the initial Mach number distribution.

In the region under consideration, $\psi \ll 1$, $dH/d\psi$ may be represented by its Taylor series expansion,

$$dH/d\psi = (dH/d\psi)_0 + \psi(d^2H/d\psi^2)_0 + \dots,$$

where L is chosen such that $(dH/d\psi)_0 = O(1)$. In view of the form of this expansion and the governing equations, it appears that a simple expansion in powers of ψ will suffice for θ and μ and therefore ϵ. That is,

$$\theta = \tilde{\theta}^{(0)}(\alpha) + \psi\tilde{\theta}^{(1)}(\alpha) + \dots, \tag{31}$$

where $\tilde{\theta}^{(0)}(\alpha) = \theta_0$, the value of θ at the corner. Similar expansions hold for μ and ϵ. On the other hand, since

$$h_\alpha \Delta\alpha = O(r\Delta\phi) = O(h_\psi \psi \Delta\phi), \quad \Delta\alpha = O(\Delta\phi)$$

and
$$h_\psi = (\rho v_\alpha)^{-1} = O(1),$$

then
$$h_\alpha = O(\psi)$$

and so h_α must have the following form of expansion,

$$h_\alpha = \psi[\tilde{h}_\alpha^{(0)} + \psi\tilde{h}_\alpha^{(1)} + \dots]. \tag{32}$$

When the above expansions are substituted into (13) and (19) and terms of like order in ψ are collected, the resulting zeroth- and first-order equations are,

$$\frac{d}{d\alpha}\left(\mu_0+\frac{\epsilon_0}{\Gamma}+\theta_0\right) = 0, \tag{33a}$$

$$\tilde{h}^{(0)}_{\alpha} = \frac{\tilde{h}^{(0)}_{\psi}}{\sin\mu_0}\frac{d}{d\alpha}\left(\frac{\epsilon_0}{\Gamma}\right), \tag{33b}$$

$$\tilde{\nu}^{(1)} = \tilde{\theta}^{(1)}+\sin\mu_0\cos\mu_0\left(\frac{dH}{d\psi}\right)_0, \tag{33c}$$

$$\tilde{h}^{(0)}_{\alpha}\tilde{\theta}^{(1)} = \tilde{h}^{(0)}_{\psi}\cos\mu_0\frac{d}{d\alpha}(\tilde{\nu}^{(1)}+\tilde{\theta}^{(1)}), \tag{33d}$$

where $\tilde{\nu}^{(1)} = \tilde{\mu}^{(1)}+\tilde{\epsilon}^{(1)}/\Gamma$ and $\tilde{h}^{(0)}_{\psi}$ is h_{ψ} evaluated at $\psi = 0$. Equation (33a) is simply a statement of the fact that the zeroth-order functions are Prandtl–Meyer solutions. This equation, combined with (23) and the fact that the characteristics are tangent to radial lines at the corner so that $\omega_0 = \pi/2$ may be used to show that

$$\frac{d}{d\alpha}\left(\frac{\epsilon_0}{\Gamma}\right) = 1. \tag{34}$$

From (7a) and (34), then,

$$d\mu_0 = -(\sin^2\mu_0+\Gamma^2\cos^2\mu_0)\,d\alpha. \tag{35}$$

Equations (33b), (33c), (33d), (34) and (35) may be used to derive a first-order, linear, non-homogeneous differential equation for $\tilde{\nu}^{(1)}+\tilde{\theta}^{(1)}$ as a function of μ_0 the solution of which is

$$\tilde{\nu}^{(1)}+\tilde{\theta}^{(1)} = \left(\frac{dH}{d\psi}\right)_0\frac{(1+\Gamma^2\cot^2\mu_0)^{(1-\Gamma^2)/4\Gamma^2}}{(\tan\mu_0)^{\frac{1}{2}}}\left\{C_1+\tfrac{1}{2}\int\frac{(\tan\mu_0)^{\frac{1}{2}}d\mu_0}{\sin^2\mu_0(1+\Gamma^2\cot^2\mu_0)^{(1+3\Gamma^2)/4\Gamma^2}}\right\}, \tag{36}$$

where C_1 is a constant. The integral in (36) may be evaluated analytically when $\gamma = 1{\cdot}4$ ($\Gamma^2 = \frac{1}{6}$), or numerically for other values of γ. Then solutions for $\tilde{\nu}^{(1)}$ and $\tilde{\theta}^{(1)}$ are found by using (36) and (33c). Finally, the equation for $\tilde{\nu}^{(1)} = \mu^{(1)}+\epsilon^{(1)}/\Gamma$ may be combined with the relationship between $\mu^{(1)}$ and $\epsilon^{(1)}/\Gamma$, found from (7a), yielding expressions for $\mu^{(1)}$ and $\epsilon^{(1)}$ as functions of μ_0.

Before the final solutions for θ and μ are written, it should be noted that they may be written in terms of $\Delta\mu_i = \mu_i-\mu_{i0}$ rather than ψ, thus illustrating the fact that they are independent of the actual distribution of μ_i (i.e. M_i). Thus, from (13a) and (7a), it can be shown that along the leading characteristic, where $\partial\theta/\partial\psi = 0$, but near the corner, where $\psi \ll 1$ and $|\mu_i-\mu_{i0}| \ll 1$,

$$\frac{1}{1-\Gamma^2}\left(\frac{dH}{d\psi}\right)_0\psi = -\frac{\cot\mu_{i0}}{\sin^2\mu_{i0}+\Gamma^2\cos^2\mu_{i0}}\Delta\mu_i. \tag{37}$$

If (37) is used to substitute for ψ in the asymptotic expansions for θ and μ, their solutions are, to first order, for $\gamma = 1{\cdot}4$,

$$\theta = \theta_0-\frac{(1-\Gamma^2)}{2}\frac{(\cot\mu_{i0})\Delta\mu_i}{(\sin^2\mu_{i0}+\Gamma^2\cos^2\mu_{i0})}\{C_1(\cot\mu_0)^{\frac{1}{2}}(1+\Gamma^2\cot^2\mu_0)^{\frac{5}{4}}$$
$$-\cot\mu_0(1+(4\Gamma^2/5)\cot^2\mu_0+\sin^2\mu_0)\}+\ldots, \tag{38a}$$

$$\mu = \mu_0 + \frac{(\cot\mu_{i0})\,\Delta\mu_i}{2(\sin^2\mu_{i0}+\Gamma^2\cos^2\mu_{i0})}\left\{C_1\frac{(1+\Gamma^2\cot^2\mu_0)^{\frac{9}{4}}}{(\cot\mu_0)^{\frac{3}{2}}} - \cot\mu_0(\sin^2\mu_0+\tfrac{4}{5}\Gamma^2)(1+\Gamma^2\cot^2\mu_0)+\ldots\right\}. \tag{38b}$$

The constant C_1 is found by imposing the condition that along the leading characteristic, $\theta = 0$. Since $\theta_0 = 0$ when $\alpha = 0$,

$$C_1 = (\cot\mu_{i0})^{\frac{1}{2}}\frac{(1+(\frac{4}{5}\Gamma^2)\cot^2\mu_{i0}+\sin^2\mu_{i0})}{(1+\Gamma^2\cot^2\mu_{i0})^{\frac{5}{4}}}. \tag{39}$$

Thus, θ and μ may be calculated from (38) and (39). As in the previous section, the solutions could be written in terms of the Mach number but, in this case, the expressions are more cumbersome. Again, the solutions give θ and μ at the intersection of a given streamline and a given left-running characteristic in terms of their values both in the incoming flow on the given streamline and at the corner on the given characteristic. Hence they are universal solutions in that they are independent of the actual Mach number distribution in a given flow. This interesting result, mentioned by Weinbaum (1966) for the solutions valid near the leading characteristic and shown here for the flow near the corner, holds only in the regions where approximate solutions may be used. In addition, it should be noted that the physical location of a given solution point does, of course, depend on the initial conditions of the problem considered.

When $M_{i0} = 1$, ($\mu_{i0} = \frac{1}{2}\pi$), it is seen that $C_1 = 0$. Then, from equations (38) it appears that as the leading characteristic is approached from downstream ($\theta_0 \to 0$, $\mu_0 \to \frac{1}{2}\pi$), $\theta \to 0$ as it should. However, in this limit, $\mu \to \frac{1}{2}\pi$ which is incorrect since no variation in the incoming flow Mach number is allowed. That is, the solutions are not uniformly valid in this limit.

An interesting mathematical point of difference between this and previous analyses may be noted. When approximate solutions for initially non-uniform irrotational or rotational flows around a corner are sought in terms of expansions in the polar variables r and ϕ, discontinuities in the first-order tangential velocity component are found on the leading characteristic (e.g. Johannesen & Meyer 1950; Pai 1954). Evidently, this is due to the fact that the leading characteristic is taken to be a zeroth-order characteristic. In the present formulation such discontinuities do not appear.

Composite solution for the case $M_{i0} > 1$

The approximate solutions in the previous two sections cover the region near the leading characteristic including the corner, and the region near the corner, including the leading characteristic. Hence it is clear that there must be a common or overlap region. In this common region the expressions must be the same; i.e. they must match, term by term. This can be proven by expanding each solution around the point at which the leading characteristic meets the corner. In terms of the Mach angles, then,

$$\left.\begin{aligned} \mu_i &= \mu_{i0}+\Delta\mu_i, \quad (\Delta\mu_i \ll 1),\\ \mu_0 &= \mu_{i0}+\Delta\mu_0, \quad (\Delta\mu_0 \ll 1),\end{aligned}\right\} \tag{40}$$

where the first expansion is used in (28) and (29) and the second expansion is used in (38) and (39). The resulting expressions for θ and μ, which are found to be the same for each expansion procedure are, for $\gamma = 1{\cdot}4$,

$$\theta = \theta_0\left[1+\frac{\Delta\mu_i}{2}\frac{\cos^2\mu_{i0}-\sin^2\mu_{i0}}{\sin\mu_{i0}\cos\mu_{i0}}\right]+\ldots, \tag{41a}$$

$$\mu = \mu_{i0}+\Delta\mu_i+\Delta\mu_0\left\{1+\frac{\Delta\mu_i}{2}\left[\cot\mu_{i0}+3\tan\mu_{i0} + 4\frac{(1-\Gamma^2)\sin\mu_{i0}\cos\mu_{i0}}{\sin^2\mu_{i0}+\Gamma^2\cos^2\mu_{i0}}\right]\right\}+\ldots. \tag{41b}$$

With the common terms known, it is possible to construct composite solutions for θ and μ uniformly valid to order $\Delta\mu_i$ near the corner and to order $\Delta\mu_0$ near the leading characteristic. The composite solutions are formed by adding the solutions valid near the leading characteristic (equations (28) and (29)) to those valid near the corner (equations (38) and (39)) and subtracting the common terms (equations (41)) (e.g. Adamson 1968).

Transonic similarity solution for the case $M_{i0} = 1$

It has been shown that the approximate solutions in both the region near the leading characteristic and the region near the corner are not uniformly valid as $M_{i0}\to 1$. This implies that for this limit there is another solution, hereafter referred to as the transonic solution, which holds in the limit, and which must match on the one hand with solutions valid near the leading characteristic and on the other hand with solutions valid near the corner. The region in which this solution holds is that region in the vicinity of the intersection of the leading characteristic with the corner, where the Mach number is only slightly larger than unity.

For this transonic analysis, it is convenient to introduce μ^*, where

$$\mu = \tfrac{1}{2}\pi-\mu^* \tag{42}$$

and where $\mu^* > 0$ and $\mu^* \ll 1$. If terms of order μ^{*2} are neglected compared to one, then the governing equations ((13) and (19)) become

$$(1-\Gamma^2)\mu^{*2}\frac{\partial\mu^*}{\partial\psi} = \frac{\partial\theta}{\partial\psi}+\mu^*\frac{dH}{d\psi}, \tag{43a}$$

$$\frac{1}{h_\psi}\frac{\partial\theta}{\partial\psi} = \frac{\mu^*}{h_\alpha}\left(\frac{\partial\theta}{\partial\alpha}+(1-\Gamma^2)\mu^{*2}\frac{\partial\mu^*}{\partial\alpha}\right), \tag{43b}$$

$$\frac{1}{h_\psi}\frac{\partial h_\alpha}{\partial\psi} = \frac{\partial\mu^*}{\partial\alpha}-\frac{\partial\theta}{\partial\alpha}, \tag{43c}$$

where (7a) was used to write the derivatives of ϵ in terms of μ^*.

Along the leading characteristic, where $\partial\theta/\partial\psi = 0$ equation (43a) may be integrated. Then, since $H = H(\psi)$ may be expanded in a Taylor series, μ_i^* may be related to ψ. Thus,

$$\mu_i^{*2} = \frac{2}{1-\Gamma^2}\left(\frac{dH}{d\psi}\right)_0\psi+\ldots \tag{44}$$

and if, as before, ψ is ordered by a characteristic length such that $(dH/d\psi)_0 = O(1)$ then $\psi = O(\mu_i^{*2})$ and $\psi \ll 1$. Hence $dH/d\psi$ may be replaced by the constant $(dH/d\psi)_0$ in (43a), consistent with the given order of approximation. Similarly, it is easily demonstrated that h_ψ may be replaced by the constant h_{ψ_0}. In addition, it is seen that θ is of order μ^{*3} from (43a) and (43b), so that the term $\partial\theta/\partial\alpha$ may be neglected compared to $\partial\mu^*/\partial\alpha$ in (43c). Finally, if the following substitutions are made for convenience,

$$\theta = (1-\Gamma^2)\theta^*, \tag{45a}$$

$$h_\alpha = h_{\psi_0} h_\alpha^*, \tag{45b}$$

$$\left(\frac{dH}{d\psi}\right)_0 = (1-\Gamma^2)\bar{H}_0', \tag{45c}$$

then (43) may be written as follows:

$$\frac{\partial}{\partial\psi}\left(\theta^* - \frac{\mu^{*3}}{3}\right) = -\mu^*\bar{H}_0', \tag{46a}$$

$$\frac{\partial}{\partial\alpha}\left(\theta^* + \frac{\mu^{*3}}{3}\right) = -h_\alpha^*\left(\bar{H}_0' - \mu^*\frac{\partial\mu^*}{\partial\psi}\right), \tag{46b}$$

$$\frac{\partial h_\alpha^*}{\partial\psi} = \frac{\partial\mu^*}{\partial\alpha}. \tag{46c}$$

Generally, elimination of two of the three dependent variables from (46), would result in a third-order differential equation for the third. However, in this case, it is possible to derive a second-order equation for μ^*. First, (46a) is differentiated with respect to α, and the $\partial\mu^*/\partial\alpha$ which occurs on the right-hand side of the equation, is replaced by $\partial h_\alpha^*/\partial\psi$ from (46c). After interchanging the order of differentiation on the left-hand side, the equation is integrated with respect to ψ and the function of integration is evaluated on the zero streamline where the Prandtl–Meyer solutions holds. When the resulting equation is subtracted from (46b), a first-order differential equation involving μ^* and h_α^* is obtained. This equation and (46c) may be combined, finally to obtain the second-order differential equation for μ^*

$$2\left(\mu^{*2}\frac{\partial\mu^*}{\partial\alpha} - \mu_0^{*2}\right)\frac{\partial}{\partial\psi}\left(\mu^*\frac{\partial\mu^*}{\partial\psi}\right) - 2\mu^*\frac{\partial\mu^*}{\partial\psi}\frac{\partial}{\partial\psi}\left(\mu^{*2}\frac{\partial\mu^*}{\partial\alpha}\right) + \left(\mu^*\frac{\partial\mu^*}{\partial\psi}\right)^2\frac{\partial\mu^*}{\partial\alpha} = 0. \tag{47}$$

Although (47) is a non-linear partial differential equation which there is little hope of solving in general, it is possible to obtain a similarity solution which satisfies the equation and boundary conditions for this problem. Before going into the similarity solution, however, it is convenient to change the independent variables from α and ψ to μ_0^* and μ_i^*, in that the boundary and matching conditions are more easily demonstrated. The following transformation is applied, then, to (47)

$$\mu_i^* = (2\bar{H}_0')^{\frac{1}{2}}\psi^{\frac{1}{2}}, \tag{48a}$$

$$\mu_0^* = \alpha, \tag{48b}$$

where (48*a*) is a result of (44), and (48*b*) a result of equation (35) for $\mu_0^* \ll 1$. Next a similarity solution of the folowing form is sought,

$$\mu^* = (\mu_i^*)^m f(\eta), \tag{49a}$$

$$\eta = \mu_0^*(\mu_i^*)^{-n}, \tag{49b}$$

and it is easily demonstrated that a similarity solution exists and the boundary condition along the leading characteristic ($\mu^* = \mu_i^*$) can be satisfied only if

$$n = m = 1.$$

Finally, then, the following equation for f may be derived from (47):

$$2\eta f(f^3-\eta^3)f'' - (f-\eta f')f'(3f^3-2\eta^3-\eta f^2 f') = 0, \tag{50}$$

where the prime denotes differentiation with respect to η. The boundary conditions on f are, as $\eta \to 0$, $f \to 1$, and as $\eta \to \infty$, $f \sim \eta$.

The asymptotic solutions to (50) are useful both for starting numerical computations and in matching procedures. They are found to be

$$f = 1 + \bar{a}\eta^{\frac{5}{2}} + \ldots \quad (\eta \to 0), \tag{51a}$$

$$f = \eta + \bar{b}\eta^{-\frac{7}{5}} + \ldots \quad (\eta \to \infty). \tag{51b}$$

A numerical solution of the non-linear differential equation (50), subject to the given boundary conditions, is difficult to carry out using standard techniques, since this is a two-point boundary value problem. However, the equation has a group property which allows one to transform the problem to an initial value problem, so that the numerical integration may be carried out quite easily (Adamson 1968). The numerical computations were carried out on The University of Michigan IBM/360 computer; values of f as a function of η were found and $\bar{a}$ and $\bar{b}$ were calculated to be

$$\bar{a} = 0{\cdot}42974, \tag{52a}$$

$$\bar{b} = 0{\cdot}40496. \tag{52b}$$

The numerical results are presented later as part of the calculation of μ and θ.

With μ^* known, it is a relatively simple matter to find θ^* and h_α^*. Thus, from (46*a*) and (46*c*), it is seen that the similarity solutions for these variables are of the form

$$\theta^* = \tfrac{1}{3}\mu^{*3} - \tfrac{1}{3}\mu_i^{*3}[g(\eta)], \tag{53a}$$

$$h_\alpha^* = \mu_i^{*2} g_1(\eta)/2\bar{H}_0'. \tag{53b}$$

In addition, if (53) and (49) are substituted into (46), the resulting ordinary differential equations may be rearranged to give g and g_1 in terms of the known f. Thus,

$$g_1 = 4(f^2 f' - \eta^2)/f(f-\eta f'), \tag{54a}$$

$$g = f + 2\eta^3 + \eta g_1/2. \tag{54b}$$

The asymptotic expansions for g and g_1 are as $\eta \to 0$

$$g = 1 + 6\bar{a}\eta^{\frac{5}{2}} + \ldots, \tag{55a}$$

$$g_1 = 10\bar{a}\eta^{\frac{3}{2}} + \ldots, \tag{55b}$$

and as $\eta \to \infty$
$$g = 2\eta^3 + \tfrac{3}{2}\eta + \tfrac{15}{22}\bar{b}\eta^{-\frac{7}{5}} + \dots, \tag{56a}$$

$$g_1 = 1 - \tfrac{7}{11}\bar{b}\eta^{-\frac{12}{5}} + \dots. \tag{56b}$$

Finally, the asymptotic solutions for μ^* and θ^* are written in terms of μ_i^* and μ_0^* for later use in matching. Equations (49) and (51) are used for μ^* and (53a), (55a) and (56a) are used for θ^*. Thus, as $\eta \to 0$

$$\mu^* = (\tfrac{1}{2}\pi) - \mu = \mu_i^* + \bar{a}(\mu_0^*)^{\frac{5}{2}}(\mu_i^*)^{-\frac{3}{2}} + \dots, \tag{57a}$$

$$\theta^* = \frac{\theta}{1-\Gamma^2} = -\bar{a}(\mu_0^*)^{\frac{5}{2}}(\mu_i^*)^{\frac{1}{2}} + \dots, \tag{57b}$$

and as $\eta \to \infty$
$$\mu^* = (\tfrac{1}{2}\pi) - \mu = \mu_0^* + \bar{b}(\mu_0^*)^{-\frac{7}{5}}(\mu_i^*)^{\frac{12}{5}} + \dots, \tag{58a}$$

$$\theta^* = \frac{\theta}{1-\Gamma^2} = \theta_0^* - (\mu_0^*/2)(\mu_i^*)^2 + \bar{b}(\mu_0^*)^{\frac{3}{5}}(\mu_i^*)^{\frac{12}{5}} + \dots, \tag{58b}$$

where
$$\tfrac{1}{3}\mu_0^{*3} = -\theta_0^*, \tag{59}$$

a result which follows from the Prandtl–Meyer solutions if $\mu_0^* \ll 1$.

Once again, θ and μ may be written in terms of μ_i^* and μ_0^* and hence are independent of the initial Mach number distribution, but h_α depends on the initial conditions through $\bar{H}_0'$, as seen in (53b).

Composite solutions valid in the region near the leading characteristic for the case $M_{i0} = 1$

Uniformly valid solutions in the region near the leading characteristic must be formed from a composite of those valid when $M > 1$ (equations (28)), and the transonic solutions, (equations (49a) and (53a)). First it is demonstrated that the two solutions match and $f_1(\alpha)$ (equations (28)) is deduced; then a composite solution is formed.

In order to match the solutions, equations (28) are expanded for $\mu_i^* \ll 1$, where again, $\mu_i^* = (\frac{1}{2}\pi) - \mu_i$. In terms of μ^* and θ^*, they become,

$$\mu^* = (\tfrac{1}{2}\pi) - \mu = \mu_i^* + \frac{f_1(\alpha)}{1-\Gamma^2}(\mu_i^*)^{-\frac{3}{2}} + \dots, \tag{60a}$$

$$\theta^* = \frac{\theta}{1-\Gamma^2} = -\frac{f_1(\alpha)}{1-\Gamma^2}(\mu_i^*)^{\frac{1}{2}} + \dots. \tag{60b}$$

These expansions are to be compared with those in (57), which are the asymptotic expansions for the transonic solutions valid near the leading characteristic (i.e. $\mu_0^* \to 0$, or $\eta \to 0$). Corresponding solutions match, term by term, if

$$\begin{aligned} f_1(\alpha) &= (1-\Gamma^2)\bar{a}(\mu_0^*)^{\frac{5}{2}} \\ &= (3)^{\frac{5}{6}}(1-\Gamma^2)^{\frac{1}{6}}\bar{a}(-\theta_0)^{\frac{5}{6}}. \end{aligned} \tag{61}$$

The change that occurs in $f_1(\alpha)$ when M_{i0} is allowed to go to unity is illustrated by comparing (61) with (29).

The proper form of the solutions valid near the leading characteristics are now equations (28) with $f_1(\alpha)$ given by (61). The composite solutions are formed by adding these solutions to the transonic solutions (equations (49a) and (53a)) and subtracting the terms common to both in the overlap region (equations (60)). The resulting expressions for μ and θ may be written in the following form:

$$\mu = (\tfrac{1}{2}\pi) - \mu_i^*\left\{f + \bar{a}\eta^{\frac{5}{2}}\left[(\sin^2\mu_i + \Gamma^2\cos^2\mu_i)(\sin\mu_i)^{\frac{1}{2}}\left(\frac{\mu_i^*}{\cos\mu_i}\right)^{\frac{3}{2}} - 1\right]\right\} + \ldots, \tag{62a}$$

$$\theta = \frac{\theta_0}{\eta^3}\left\{g - f^3 + 3\bar{a}\eta^{\frac{5}{2}}\left[\left(\frac{\sin\mu_i\cos\mu_i}{\mu_i^*}\right)^{\frac{1}{2}} - 1\right]\right\} + \ldots. \tag{62b}$$

These solutions are uniformly valid to order $(\mu_0^*)^{\frac{5}{2}}$. With μ and thus ϵ known, all the thermodynamic functions and u_α and v_α may be calculated. Comparison of (62) with exact numerical results is given later.

Composite solutions valid in the region near the corner for the case $M_{i0} = 1$

Uniformly valid solutions are formed from solutions valid near the corner, for $M > 1$ and the transonic solutions. Again, matching is demonstrated first, and then composite solutions are formed.

The solutions valid near the corner for $M > 1$ are essentially those given in (38), with one important difference. In the present case, $\psi(dH/d\psi)_0$ is replaced by μ_i^{*2} as in (44) rather than by $\Delta\mu_i$ as given in (37), when the solutions for μ and θ are considered. If the resulting solutions are expanded, then, for $\mu_0^* \ll 1$,

$$\mu^* = (\tfrac{1}{2}\pi) - \mu = \mu_0^* + \frac{(\mu_i^*)^2}{4}\left\{\frac{C_1}{(\mu_0^*)^{\frac{3}{2}}} - \mu_0^*\left(1 + \frac{4\Gamma^2}{5}\right)\right\} + \ldots, \tag{63a}$$

$$\theta^* = \frac{\theta}{1-\Gamma^2} = \frac{\theta_0}{1-\Gamma^2} + \frac{(\mu_i^*)^2}{4}\{C_1(\mu_0^*)^{\frac{1}{2}} - 2\mu_0^*\} + \ldots. \tag{63b}$$

These equations are to be compared with (58), which are the asymptotic expansions for the transonic solutions, valid near the corner (i.e. $\mu_i^* \to 0$, or $\eta \to \infty$). First, consideration of the expansions for θ^* indicates that if matching is to occur at least up to terms of order $(\mu_i^*)^2$, $C_1 = 0$. This is consistent with the expression derived for C_1 previously (39), in that as $\mu_{i0} \to \frac{1}{2}\pi$ or $\mu_{i\,0}^* \to 0$, $C_1 \to 0$. The fact that the term of order $(\mu_i^*)^2\mu_0^*$ is not matched in the expression for μ^* is due simply to the fact that terms of order $(\mu^*)^2$ have been neglected compared to one in this analysis, and the zeroth-order term is μ_0^*. Since θ_0 is of order $(\mu_0^*)^3$, the $(\mu_i^*)^2\mu_0^*$ term is important in the expression for θ. In addition, it can be shown that if terms of order $(\mu^*)^2$ are retained in the transonic equations then the similarity solution for μ^* is,

$$\mu^* = \mu_i^* f^{(1)}(\eta) + (\mu_i^*)^3 f^{(3)}(\eta) + \ldots$$

and the second term could match the $(\mu_i^*)^2\mu_0^*$ term.

Although the above arguments explain the matching of all the terms shown for the 'outer' corner solutions (63), it is seen from (58) that there are terms in the transonic 'inner' solutions that do not exist in the outer solution as presently constructed. That is, since $\mu_i^* = K\psi^{\frac{1}{2}}$, where K is a constant, the inner (58a) and outer (63a) equations for μ^* are, in terms of ψ,

$$\mu^* = \mu_0^* + \bar{b}K^{\frac{12}{5}}\psi^{\frac{6}{5}}(\mu_0^*)^{-\frac{7}{5}} + \ldots,$$

$$\mu^* = \mu_0^* - \frac{K^2}{4}\left(1 + \frac{4\Gamma^2}{5}\right)\psi\mu_0^* + \ldots,$$

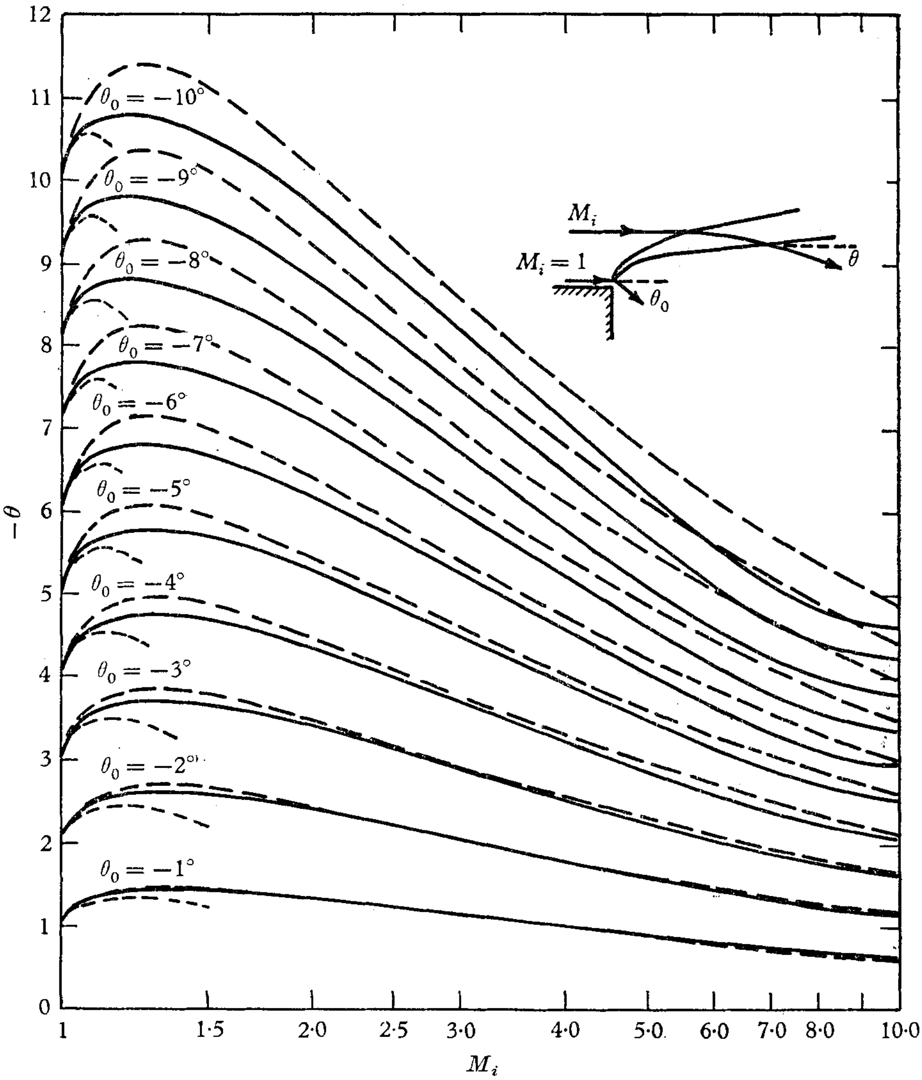

FIGURE 4. Comparison of flow deflexion angle calculated from solutions valid near the leading characteristic (62b) and valid near the corner (67b) with exact numerical solutions (Weinbaum 1966). $\gamma = 1{\cdot}4$. ——, exact numerical solution. Approximate solutions: — —, valid near leading characteristic; - - -, valid near corner.

where, again, $C_1 = 0$. It is clear that since a simple expansion in integer powers of ψ was postulated in the outer region, as a result of the form of $dH/d\psi$, no counterpart of the inner term of order $\psi^{\frac{6}{5}}$ exists presently. On the other hand it is seen that non-integer powers of ψ could be considered as long as the terms involved satisfy the homogeneous parts of the governing equations. That is, as a result of the matching procedure, it is evident that the outer solutions for this case should be of the form,

$$\mu = \mu_0 + \psi\tilde{\mu}^{(1)} + \psi^{\frac{6}{5}}\tilde{\mu}^{(\frac{6}{5})} + \dots \quad (64)$$

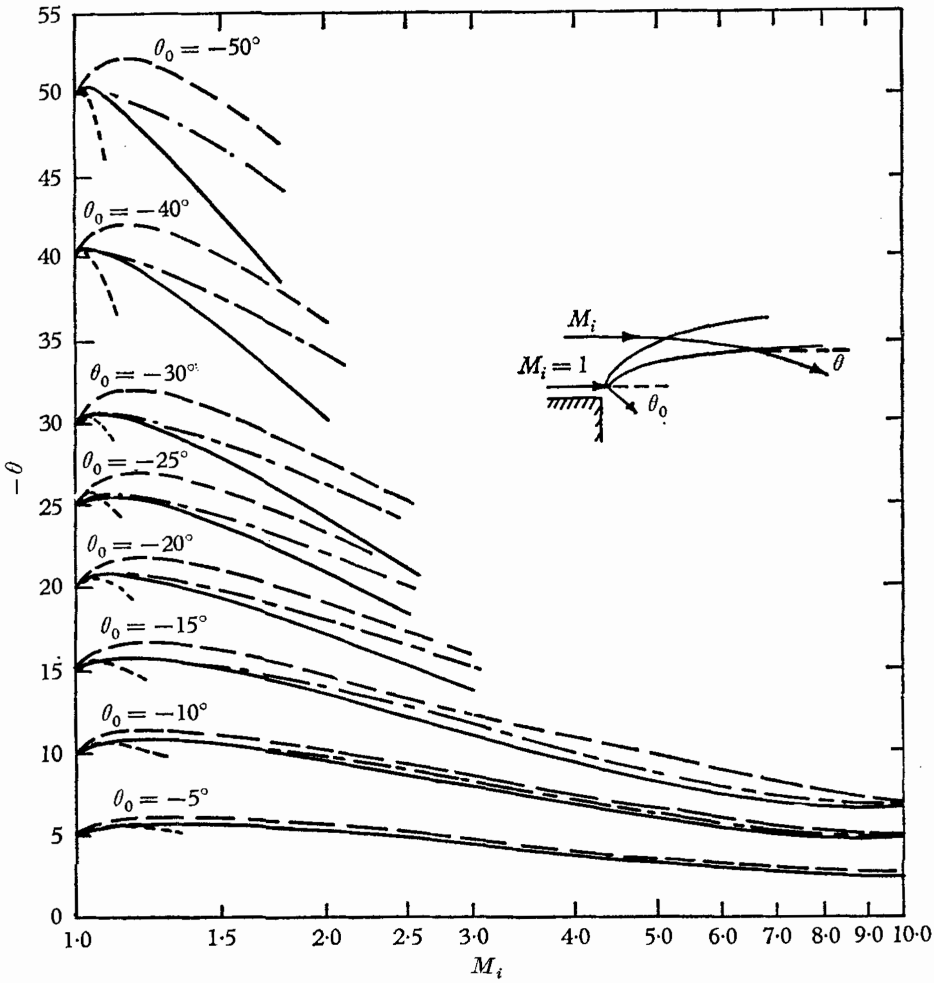

FIGURE 5. Comparison of flow deflexion angle calculated from solutions valid near the leading characteristic (62*b*) and valid near the corner (67*b*) with exact numerical solutions for two different initial Mach number distributions (Weinbaum 1966). $\gamma = 1{\cdot}4$. ——, —·—, exact numerical solutions with different initial Mach number distributions. Approximate solutions: — — —, valid rear leading characteristic; - - -, valid near corner.

The addition of terms due to matching requirements is a very common occurrence in the method of matched asymptotic expansions. Physically, the inclusion of this term implies that since the flow is very nearly sonic, the incoming flow near the corner can influence the flow at the corner.

When expansions of the form shown in (64) are substituted into the governing equations, solutions are found easily. Thus, for the flow near the corner but far enough away from the leading characteristic that $M > 1$,

$$\mu = \mu_0 - \frac{\psi}{2(1-\Gamma^2)}\left(\frac{dH}{d\psi}\right)_0 \{C_1(1+\Gamma^2\cot^2\mu_0)^{\frac{9}{4}}(\cot\mu_0)^{-\frac{3}{2}} - \cot\mu_0(1+\Gamma^2\cot^2\mu_0)((4\Gamma^2/5)+\sin^2\mu_0)\} - \frac{\psi^{\frac{6}{5}}}{(1-\Gamma^2)} D_1[1+\Gamma^2\cot^2\mu_0]^{\frac{5}{2}}(\cot\mu_0)^{-\frac{7}{5}} + \ldots, \tag{65a}$$

$$\theta = \theta_0 + \frac{\psi}{2}\left(\frac{dH}{d\psi}\right)_0 \left\{C_1(\cot\mu_0)^{\frac{1}{2}}(1+\Gamma^2\cot^2\mu_0)^{\frac{5}{4}} - \cot\mu_0\left(1+\frac{4\Gamma^2}{5}\cot^2\mu_0+\sin^2\mu_0\right)\right\} + \psi^{\frac{6}{5}}D_1(1+\Gamma^2\cot^2\mu_0)^{\frac{3}{2}}(\cot\mu_0)^{\frac{3}{5}} + \ldots, \tag{65b}$$

where D_1 is a constant and again $\gamma = 1{\cdot}4$. If ψ is replaced by μ_i^{*2}, according to (48a), and equations (65) are expanded for $\mu_0^* \ll 1$ and matched with (58), it is seen that

$$C_1 = 0, \quad D_1 = \bar{b}(1-\Gamma^2)\left[\frac{2(dH/d\psi)_0}{1-\Gamma^2}\right]^{\frac{6}{5}}. \tag{66}$$

The composite solutions for this region are formed by adding (65) (with the constants as in (66)) to the transonic solutions, (49a) and (53a), and subtracting the terms common to both solutions, (58). They may be written as follows:

$$\mu = (\pi/2) - \mu_i^*\left\{f - \frac{\mu_i^*}{4}\cot\mu_0(1+\Gamma^2\cot^2\mu_0)\left(\frac{4\Gamma^2}{5}+\sin^2\mu_0\right) + \bar{b}\eta^{-\frac{7}{5}}\left[\left(\frac{\mu_0^*}{\cot\mu_0}\right)^{\frac{7}{5}}(1+\Gamma^2\cot^2\mu_0)^{\frac{5}{2}} - 1\right]\right\} + \ldots, \tag{67a}$$

$$\theta = \frac{\theta_0}{\eta^3}\left\{g - f^3 + \tfrac{3}{4}\eta\left[\left(1+\frac{4\Gamma^2}{5}\cot^2\mu_0+\sin^2\mu_0\right)\left(\frac{\cot\mu_0}{\mu_0^*}\right) - 2\right] - 3\bar{b}\eta^{\frac{3}{5}}\left[\left(\frac{\cot\mu_0}{\mu_0^*}\right)^{\frac{3}{5}}(1+\Gamma^2\cot^2\mu_0)^{\frac{3}{2}} - 1\right]\right\} + \ldots, \tag{67b}$$

where, again, $\Gamma^2 = \frac{1}{6}$. These equations are uniformly valid to order $(\mu_i^*)^{\frac{12}{5}}$. Again, with μ known, the thermodynamic properties and u_α and v_α may be found easily.

Comparison of asymptotic solutions with exact numerical computations

In figures (4) and (5), the composite solutions for θ valid near the leading characteristic (62b) and valid near the corner (67b) are compared with exact numerical solutions given by Weinbaum (1965, 1966) for rotational flow around a corner. θ is plotted versus M_i for various values of $(-\theta_0)$, which corresponds to plotting the variation of θ along a given characteristic ($-\theta_0 = \text{const.}$). The method of rotational characteristics was used for the exact numerical computations. In figure 5, exact numerical solutions are shown for two different initial

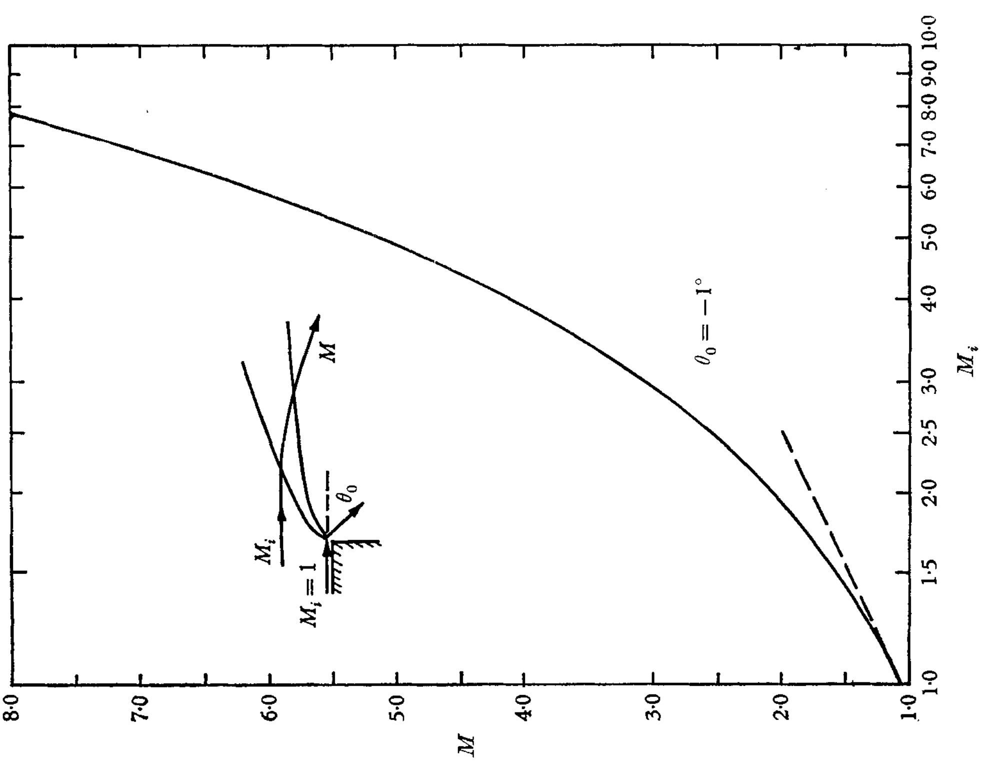

FIGURE 6*a*. For legend see next page.

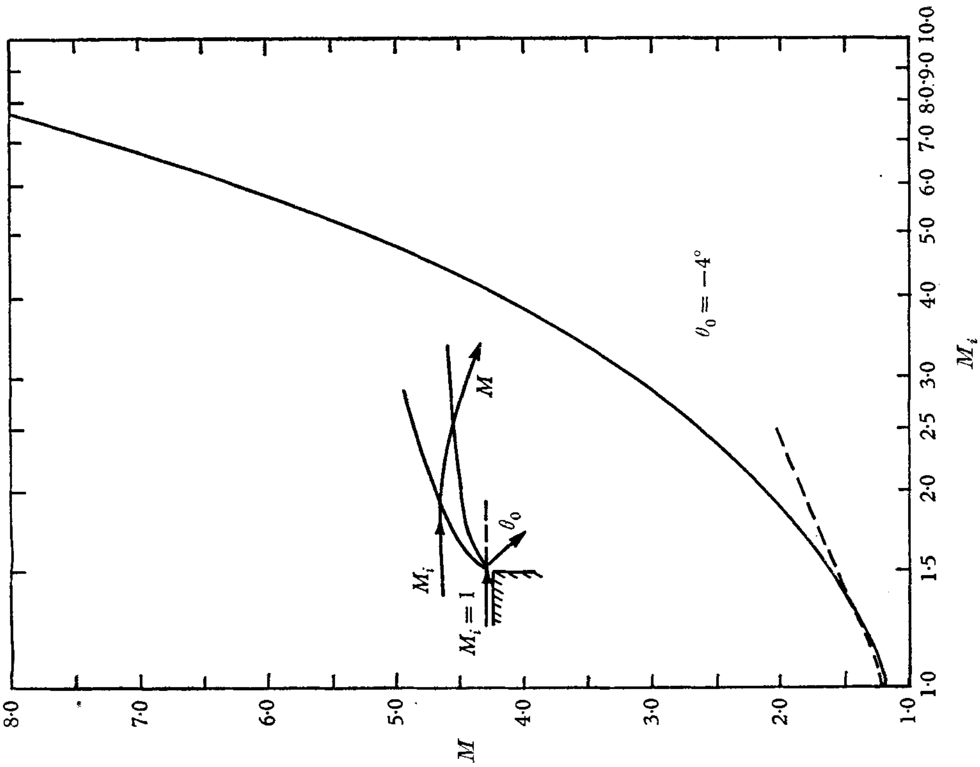

FIGURE 6*b*. For legend see next page.

Mach number distributions to illustrate the fact that near the leading characteristic ($|\theta_0| \ll 1$) and near the corner ($M_i - 1 \ll 1$), the solutions merge and thus are independent of the initial Mach number distribution as indicated by the approximate solutions. It can be seen that for $|\theta_0| \ll 1$, the solutions valid near the leading characteristic are essentially identical with the exact numerical

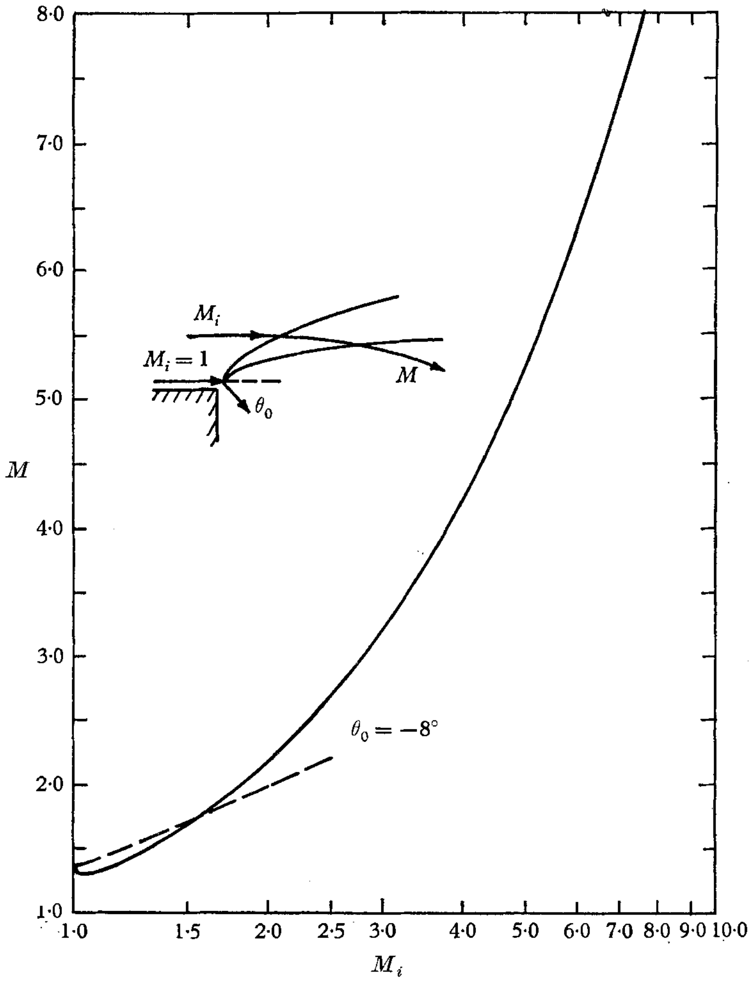

FIGURE 6*c*

FIGURE 6 (*a*–*c*). Mach number distribution along the characteristic corresponding to a given θ_0 for solutions valid near the leading characteristic (62*a*) and near the corner (67*a*) with $M = (\sin \mu)^{-1}$. Approximate solutions: ———, valid near the leading characteristic; — — —, valid near the corner.

solution, but that as $|\theta_0|$ increases, the error in the maximum value of $|\theta|$ becomes larger and larger. In addition, for $M_i - 1 \ll 1$, the solution valid near the corner agrees with the exact numerical solution for all $|\theta_0|$, although above $-\theta_0 = 20°$, the agreement is not consistently good. It seems evident that since the method of rotational characteritics involves a finite mesh size, the values calculated for θ become less precise as the corner is approached, for large $|\theta_0|$; in this region the analytical expressions presented here are probably more accurate.

In figures 6*a*, *b* and *c*, typical variations in the Mach number are given for the two approximate solutions, (62*a*) and (67*a*). Exact numerical calculations were not available for comparison. These figures show that the solution valid near the leading characteristic gives the proper slope for the Mach number distribution at the corner, only when $|\theta_0| \ll 1$.

5. Discussion

The transonic similarity solution in the case where $M_{i0} = 1$ allows one to form composite solutions from which numerical values for θ and μ may be found without recourse to 'matching' with numerical solutions near the sonic region, as has been done previously. Since composite solutions were found both in the region near the leading characteristic and in the region near the corner, the question arises as to whether it is possible to construct one over-all composite solution uniformly valid in both regions. It can be shown that this is not possible unless higher order approximations are found for μ.

It may be noted that the solutions found for θ and μ are not uniformly valid as $M_1 \to \infty$. This could be remedied by consideration of a hypersonic approximation where $\mu_i \ll 1$ and the inclusion of the resulting hypersonic solutions in the composite solutions.

Finally, it is interesting to note that the transonic similarity variable, η, has a form different from that found previously by Vaglio-Laurin (1960) for the problem of a subsonic stream being accelerated around a corner, to supersonic speeds. A simple calculation shows that in terms of Cartesian co-ordinates, $\eta = O(xy^{-\frac{3}{2}})$; on the other hand Vaglio-Laurin found a similarity variable, $\zeta = O(xy^{-\frac{5}{4}})$. The reason for this difference lies, of course, in the fact that different initial conditions are considered in the two problems.

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